

# Arrangements of pseudocircles on surfaces

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## Abstract

A pseudocircle is a simple closed curve on some surface. Arrangements of pseudocircles were introduced by Grünbaum, who defined them as collections of pseudocircles that pairwise intersect in exactly two points, at which they cross. There are several variations on this notion in the literature, one of which requires that no three pseudocircles have a point in common. Working under this definition, Ortner proved that an arrangement of pseudocircles is embeddable into the sphere if and only if all of its subarrangements of size at most 4 are embeddable into the sphere. Ortner asked if an analogous result held for embeddability into a compact orientable surface  $\Sigma_g$  of genus  $g > 0$ . In this paper we answer this question, under an even more general definition of an arrangement, in which the pseudocircles in the collection are not required to intersect each other, or that the intersections are crossings: it suffices to have one pseudocircle that intersects all other pseudocircles in the collection. We show that under this more general notion, an arrangement of pseudocircles is embeddable into  $\Sigma_g$  if and only if all of its subarrangements of size at most  $4g + 5$  are embeddable into  $\Sigma_g$ , and that this can be improved to  $4g + 4$  under the concept of an arrangement used by Ortner. Our framework also allows us to generalize this result to arrangements of other objects, such as arcs.

## 1 Introduction

This work is motivated by a question posed in [14]. In that paper, Ortner proved that an arrangement of pseudocircles is embeddable into the sphere if and only if all of its subarrangements of size at most 4 are embeddable into the sphere. Ortner asked if an analogous result held for embeddability into a compact orientable surface  $\Sigma_g$  of genus  $g > 0$ . We answer this question positively, under an even more general definition of an arrangement of pseudocircles than the one considered in [14]:

**Theorem 1.** *An arrangement of pseudocircles is embeddable into  $\Sigma_g$  if and only if all of its subarrangements of size at most  $4g + 5$  are embeddable into  $\Sigma_g$ .*

As we will see, for the arrangements investigated in [14] (what we will call *strong* arrangements), the size bound  $4g + 5$  can be improved to  $4g + 4$ .

We show that Theorem 1 follows as a consequence of a more general result on the genera of subgraphs of an embedded graph (namely Main Theorem in Section 2). This connection is based

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on the embedded graph naturally induced by an arrangement of pseudocircles. As we show at the end of this section, Theorem 1 can be equivalently formulated by saying that if the embedded graph induced by an arrangement of pseudocircles has genus greater than  $g$  for some  $g \geq 0$  (we recall the definition of the genus of an embedded graph in Section 1.2), then there is a subarrangement of size at most  $4g + 5$  whose induced embedded graph has genus greater than  $g$ .

Before we arrive to this equivalent formulation, we need to review the concept of an arrangement of pseudocircles (Section 1.1), as well as the notion of the embeddability of an arrangement into a surface (Section 1.2).

## 1.1 Arrangements of pseudocircles

A *pseudocircle* is a simple closed curve on a surface. There exist several variations on the definition of an arrangement of pseudocircles in the literature. These objects were introduced by Grünbaum in [7] (he called them *arrangements of curves*), who required that any two pseudocircles in the collection intersect each other in exactly two points, at which they cross. Under Grünbaum's definition, arrangements of pseudocircles generalize arrangements of circles, in the same way as arrangements of pseudolines generalize arrangements of lines.

Sometimes the pseudocircles in the collection are not required to intersect each other (in [15], an arrangement in which every two pseudocircles intersect is called *complete*). This relaxed definition is used for instance in [8], where Kang and Müller showed (among other results) that every arrangement of at most four pseudocircles in the plane is isomorphic to an arrangement of circles (see also [10, 11]). In addition, sometimes tangential intersections between pseudocircles are allowed. Grünbaum himself proposed this relaxed notion where tangential intersections (or *osculations*) are possible, leading to the concept of a *weak arrangement of curves*. This more general notion is adopted for instance in [1], where Agarwal et al. gave an upper bound on the number of empty lenses in arrangements of pseudocircles, and derived several important applications of this result. Moreover, in the combinatorial formalism of arrangements given in [9], Linhart and Ortner allow pseudocircles to intersect each other more than twice.

The definition used in [14] is in line with the original concept introduced by Grünbaum, with the additional condition that no three pseudocircles meet at a common point. In [14], Ortner defines an arrangement of pseudocircles as a finite collection of pseudocircles in some compact orientable surface, such that:

- (i) no three pseudocircles meet each other at the same point;
- (ii) each intersection point between pseudocircles is a crossing, rather than tangential; and
- (iii) each pair of pseudocircles intersects exactly twice.

We call these collections *strong arrangements of pseudocircles*, to distinguish them from a more general version that we present below.

The motivation behind the version we introduce below is that we realized that our results hold in this more general setting. We need not assume Conditions (i) and (ii). Moreover, we do not need the full strength of (iii), where it is required that every pair of pseudocircles intersect each other: it suffices to ask that there is a pseudocircle intersected by all the other pseudocircles in the collection.

**Definition 1.** *An arrangement of pseudocircles is a finite collection of pseudocircles in some compact orientable surface (the host surface of the arrangement) that pairwise intersect a finite number of times (possibly zero), and such that there exists a pseudocircle that is intersected by all the other pseudocircles in the collection. A pseudocircle with this property (it need not be unique) is an anchor of the arrangement.*

This is the definition that we adopt in this paper. Clearly, every strong arrangement is also an arrangement according to this definition. A natural generalization of this definition would be to drop the requirement that one pseudocircle is intersected by all the others. However, as we discuss in Section 8 (and, as it was pointed out in [14]), without some minimal requirement of this form, no result along the lines of Theorem 1 holds.

## 1.2 Embeddability of an arrangement of pseudocircles into a surface

Theorem 1 is a statement about the embeddability of an arrangement of pseudocircles into a surface. Since an arrangement is by definition already embedded on a surface, the notion of its embeddability into another surface must be clarified. This concept is based on the isomorphism between arrangements of pseudocircles.

An arrangement of pseudocircles  $\Gamma$  can be naturally regarded as an embedded graph, whose vertices are the points where the pseudocircles intersect each other. Following [14], this embedded graph is the *arrangement graph* of  $\Gamma$ .

We emphasize that an arrangement graph is an *embedded graph*, that is, an abstract graph (a combinatorial entity with vertices and edges) with a *fixed* embedding on some surface. To continue with our discussion on the embeddability of an arrangement into a surface, we need to recall when two embedded graphs are isomorphic. To proceed, we first remind the reader that the *rotation* around a vertex  $v$  in an embedded graph  $G$  is a cyclic permutation of the edges incident with  $v$ ; this cyclic rotation records the clockwise order in which these edges leave  $v$  in the embedding.

Suppose that  $G$  is an embedded graph (on some surface), with vertex set  $V$  and edge set  $E$ , and  $G'$  is an embedded graph (on some surface), with vertex set  $V'$  and edge set  $E'$ . Then (the embedded graphs)  $G$  and  $G'$  are *isomorphic* if there is a mapping  $\phi : V \cup E \rightarrow V' \cup E'$  that is a graph isomorphism when  $G$  and  $G'$  are regarded as abstract graphs, and in addition the following holds: if the rotation in  $G$  of vertex  $v$  is  $(e_1 e_2 \cdots e_m)$ , then the rotation of  $\phi(v)$  in  $G'$  is  $(\phi(e_1) \phi(e_2) \cdots \phi(e_m))$ . Thus two embedded graphs are isomorphic if their underlying abstract graphs have an isomorphism that preserves and reflects not only the structure of the graphs but also their embeddings.

*Remark.* Throughout this paper, whenever we have two embedded graphs  $G$  and  $G'$ , and mention they are isomorphic, it is tacitly understood that they are isomorphic as embedded graphs, and not only (the weaker, implied fact) that their underlying abstract graphs are isomorphic.

We are now ready to recall when two arrangements of pseudocircles are isomorphic. Let  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$  and  $\Delta = \{\delta_1, \delta_2, \dots, \delta_n\}$  be arrangements of pseudocircles (note they have the same size). Let  $G$  and  $G'$  be the arrangement graphs of  $\Gamma$  and  $\Delta$ , respectively. Then  $\Gamma$  and  $\Delta$  are isomorphic arrangements if there is an isomorphism from  $G$  to  $G'$  that maps the pseudocircles in  $\Gamma$  to the pseudocircles in  $\Delta$ . Formally,  $\Gamma$  and  $\Delta$  are *isomorphic* if there is an isomorphism  $\phi : V \cup E \rightarrow V' \cup E'$  from  $G$  to  $G'$ , and a permutation  $\rho(1) \rho(2) \cdots \rho(n)$  of  $1 2 \cdots n$  such that the following holds: if the cycle in  $G$  corresponding to the pseudocircle  $\gamma_i$  is  $v_0 e_1 v_1 \cdots e_m v_0$ , then the cycle in  $G'$  corresponding to the pseudocircle  $\delta_{\rho(i)}$  is  $\phi(v_0) \phi(e_1) \phi(v_1) \cdots \phi(e_m) \phi(v_0)$ .

At an informal level, this reflects the intuitive notion that two arrangements of pseudocircles are isomorphic if after one removes the whole host surface except for a very thin strip around each edge, and a very small disk around each vertex, the arrangements are undistinguishable.

**Definition 2.** *An arrangement of pseudocircles  $\Gamma$  is embeddable into  $\Sigma_g$  if there is an arrangement  $\Delta$  isomorphic to  $\Gamma$  such that the host surface of  $\Delta$  is  $\Sigma_g$ .*

We prove Theorem 1 under an equivalent form we give below, which is given in terms of the genus of embedded graphs. We refer the reader to [13] for basic concepts on graph embeddings, such as the facial walks ([13, Sec. 4.1]) and the genus ([13, Eq. (4.2)]) of an embedded graph.

For this discussion we recall that if  $G$  is an embedded graph with vertex set  $V$ , edge set  $E$ , and set of facial walks  $\mathcal{W}$ , then the *genus*  $\text{gen}(G)$  of  $G$  is  $\text{gen}(G) := (1/2)(2 - |V| + |E| - |\mathcal{W}|)$ . The essential property of the genus of an embedded graph that we will use is that  $G$  is isomorphic to a graph embedded in  $\Sigma_g$  if and only if  $\text{gen}(G) \leq g$ .

Let  $\Gamma$  be an arrangement of pseudocircles, and let  $G$  be its arrangement graph. It follows immediately from Definition 2 that  $\Gamma$  is embeddable into  $\Sigma_g$  if and only if  $G$  is isomorphic to a graph embedded in  $\Sigma_g$ . Now from our previous remark, this last condition holds if and only if  $\text{gen}(G) \leq g$ . Thus we obtain that  $\Gamma$  is embeddable into  $\Sigma_g$  if and only if  $\text{gen}(G) \leq g$ . Equivalently,  $\Gamma$  is *not* embeddable into  $\Sigma_g$  if and only if  $\text{gen}(G) > g$ .

With this observation in hand, we note that Theorem 1 can be equivalently interpreted by saying that if  $\Gamma$  is not embeddable into  $\Sigma_g$ , then  $\Gamma$  has a subarrangement  $\Gamma'$  of bounded size (at most  $4g + 5$ ) that already witnesses this non-embeddability; that is, the arrangement graph  $G'$  of  $\Gamma'$  satisfies  $\text{gen}(G') > g$ . We now write this equivalent formulation of Theorem 1 formally, as this is the version under which we work for the rest of the paper.

**Theorem 1** (Equivalent form). *Let  $\Gamma$  be an arrangement of pseudocircles with arrangement graph  $G$ , and let  $g \geq 0$  be an integer. Then  $\text{gen}(G) > g$  if and only if  $\Gamma$  has a subarrangement  $\Gamma'$ , with  $|\Gamma'| \leq 4g + 5$ , such that the arrangement graph  $G'$  of  $\Gamma'$  satisfies  $\text{gen}(G') > g$ .*

## 2 Clusters of graphs, the Main Theorem, and Proof of Theorem 1

As we came up with the proof of Theorem 1, we realized that our arguments held in a more general setting, and so we ended up obtaining it as a consequence of a more general result. In this section we state this result (the Main Theorem of this paper), and show that Theorem 1 follows as a corollary.

In its equivalent formulation given at the end of Section 1, Theorem 1 can be interpreted as saying that if an embedded graph  $G$  with  $\text{gen}(G) > g$  can be decomposed into a collection  $\mathcal{C}$  of edge-disjoint cycles, where one cycle in  $\mathcal{C}$  intersects all the other cycles in  $\mathcal{C}$ , then there is a subcollection  $\mathcal{C}'$  of  $\mathcal{C}$ , with  $|\mathcal{C}'| \leq 4g + 5$ , such that  $\text{gen}(\bigcup_{C \in \mathcal{C}'} C) > g$ . When we proved this statement, we realized that our arguments did not depend on the assumption that the elements of  $\mathcal{C}$  were cycles; we only needed their connectedness, and the property that some element in  $\mathcal{C}$  intersects all the other elements of  $\mathcal{C}$ .

This led us to the following concept. A collection  $\mathcal{H}$  of pairwise edge-disjoint connected graphs simultaneously embedded in a surface is a *cluster of graphs* if there is a graph  $H$  in  $\mathcal{H}$  (an *anchor* of  $\mathcal{H}$ ) that intersects every graph in  $\mathcal{H}$ .

The arrangement graph associated to an arrangement of pseudocircles can thus be naturally regarded as (the union of) a cluster of graphs: each pseudocircle corresponds to a cycle in the cluster, where the cycle that corresponds to an anchor pseudocircle is an anchor of the cluster.

Our main result in this paper is the following statement, which is thus a generalization of Theorem 1. Throughout this paper, if  $\mathcal{H}$  is a family of graphs embedded on the same surface (such as a cluster), we use  $\bigcup \mathcal{H}$  to denote the embedded graph that is the union of the elements of  $\mathcal{H}$ .

**Main Theorem** (Implies Theorem 1). *Let  $\mathcal{H}$  be a cluster of graphs such that  $\text{gen}(\bigcup \mathcal{H}) > g$ , for some  $g \geq 0$ . Then there is an  $\mathcal{H}_g \subseteq \mathcal{H}$  with  $|\mathcal{H}_g| \leq 4g + 5$ , such that  $\text{gen}(\bigcup \mathcal{H}_g) > g$ .*

In Section 3 we state two lemmas and show that they imply the Main Theorem. The rest of the paper is then almost entirely devoted to the proofs of these lemmas.

We close this section by showing that Theorem 1 is an easy consequence of the Main Theorem. From the previous discussion this could be seen as a mere formality, but we write it for completeness.

*Proof of Theorem 1.* As we have mentioned, we will prove Theorem 1 in its equivalent formulation given at the end of Section 1. The “only if” part is trivial: if a subgraph  $G'$  of an embedded graph  $G$  satisfies  $\text{gen}(G') > g$ , then obviously  $\text{gen}(G) > g$ .

For the “if” part, let  $\Gamma = \{\gamma, \gamma_1, \dots, \gamma_n\}$  be an arrangement of pseudocircles, where  $\gamma$  is an anchor of  $\Gamma$ . Let  $G$  be the arrangement graph of  $\Gamma$ , and suppose  $\text{gen}(G) > g$  for some  $g \geq 0$ . Now let  $C$  be the cycle in  $G$  induced by  $\gamma$ , and let  $C_i$  be the cycle in  $G$  induced by  $\gamma_i$ , for  $i = 1, 2, \dots, n$ .

Then clearly  $\mathcal{C} = \{C, C_1, \dots, C_n\}$  is a cluster of graphs with anchor  $C$ . By the Main Theorem, there exists a  $\mathcal{C}_g \subseteq \mathcal{C}$ , with  $|\mathcal{C}_g| \leq 4g + 5$ , such that  $\text{gen}(\bigcup \mathcal{C}_g) > g$ . Now let  $\Gamma'$  be the subcollection of  $\Gamma$  that consists of the pseudocircles that induce the cycles in  $\mathcal{C}_g$ . Then  $\Gamma'$  satisfies the required conditions, since  $|\Gamma'| = |\mathcal{C}_g| \leq 4g + 5$ ,  $\bigcup \mathcal{C}_g$  is the arrangement graph of  $\Gamma'$ , and  $\text{gen}(\bigcup \mathcal{C}_g) > g$ .  $\square$

As we have already mentioned, the size bound  $4g + 5$  in Theorem 1 can be slightly refined (to  $4g + 4$ ) for the class of arrangements considered in [14]. This improvement relies not only on the Main Theorem, but on its proof. Thus we prove this refinement of Theorem 1 in the next section, immediately after the proof of the Main Theorem (see Remark at the end of Section 3.1).

### 3 Reducing the Main Theorem to two lemmas

As we shall see shortly, the Main Theorem follows easily by an induction based on the following statement. We note that if  $H$  is an anchor of a cluster  $\mathcal{H}$ , then in particular  $H$  is an embedded graph (subgraph of the embedded graph  $\bigcup \mathcal{H}$ ), and as such,  $H$  has a genus. We encourage the reader to follow our own custom, which is to informally think of our next statement as “at most 4 graphs of the cluster need to be added to the anchor, to obtain a graph whose genus is greater than the genus of the anchor”.

**Theorem 2.** *Let  $\mathcal{H}$  be a cluster of graphs with anchor  $H$ . Suppose that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ . Then there is a subcollection  $\overline{\mathcal{H}} \subseteq \mathcal{H}$ , with  $H \in \overline{\mathcal{H}}$  and  $|\overline{\mathcal{H}}| \leq 5$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \overline{\mathcal{H}})$ .*

In Section 3.2 we state two lemmas and show that they imply Theorem 2. Before we proceed to that, we show that the Main Theorem follows from this statement.

### 3.1 The Main Theorem follows from Theorem 2

*Proof of the Main Theorem.* Assuming Theorem 2, we prove the Main Theorem by induction on  $g$ , for a fixed cluster of graphs  $\mathcal{H}$  with anchor  $H$ .

For the base case  $g = 0$  the assumption is that  $\text{gen}(\bigcup \mathcal{H}) > 0$ . If  $\text{gen}(H) > 0$  then we are done by taking  $\mathcal{H}_0 := \{H\}$ , and so we may assume that  $\text{gen}(H) = 0$ . In this case we apply Theorem 2, to obtain a subcollection  $\overline{\mathcal{H}} \subseteq \mathcal{H}$ , with  $H \in \overline{\mathcal{H}}$  and  $|\overline{\mathcal{H}}| \leq 5$  such that  $\text{gen}(H) < \text{gen}(\bigcup \overline{\mathcal{H}})$ . Thus in particular  $\text{gen}(\bigcup \overline{\mathcal{H}}) > 0$ , and so we are done by setting  $\mathcal{H}_0 := \overline{\mathcal{H}}$ .

We now suppose that the Main Theorem holds for  $g = h$  for some  $h \in \{0, 1, \dots, \text{gen}(\bigcup \mathcal{H}) - 2\}$ , and show that then it holds for  $g = h + 1$ . (Note that the largest value of  $g$  for which the statement of the Main Theorem makes sense is  $g = \text{gen}(\bigcup \mathcal{H}) - 1$ ).

The assumption that the statement holds for  $g = h$  means that there is a subcollection  $\mathcal{H}_h$  of  $\mathcal{H}$  with  $|\mathcal{H}_h| \leq 4h + 5$  such that  $\text{gen}(\bigcup \mathcal{H}_h) > h$ . If  $\text{gen}(\bigcup \mathcal{H}_h) > h + 1$  then we are done by setting  $\mathcal{H}_{h+1} := \mathcal{H}_h$ , so we may assume that  $\text{gen}(\bigcup \mathcal{H}_h) = h + 1$ . Note that since  $h \leq \text{gen}(\bigcup \mathcal{H}) - 2$ , it follows that  $\text{gen}(\bigcup \mathcal{H}) > h + 1$ .

Let  $K := \bigcup \mathcal{H}_h$ , and let  $\mathcal{K}$  be the collection  $\{K\} \cup (\mathcal{H} \setminus \mathcal{H}_h)$ . We claim that  $\mathcal{K}$  is an anchor of graphs with cluster  $K$ . To see this, note that every graph in  $\mathcal{H}$  intersects  $H$ , and so in particular every graph in  $\mathcal{H} \setminus \mathcal{H}_h$  intersects  $H$ . Since  $H$  is contained in  $K$ , it follows that every graph in  $\mathcal{H} \setminus \mathcal{H}_h$  intersects  $K$ . This proves the claim.

Note that  $\bigcup \mathcal{K} = \bigcup \mathcal{H}$ . Thus  $\text{gen}(\bigcup \mathcal{K}) = \text{gen}(\bigcup \mathcal{H}) > h + 1$ . Since  $\text{gen}(K) = \text{gen}(\bigcup \mathcal{H}_h) = h + 1$ , it follows that  $\text{gen}(K) < \text{gen}(\bigcup \mathcal{K})$ . Thus by Theorem 2 there is a subcollection  $\overline{\mathcal{K}} \subseteq \mathcal{K}$ , with  $K \in \overline{\mathcal{K}}$  and  $|\overline{\mathcal{K}}| \leq 5$ , such that  $\text{gen}(\bigcup \overline{\mathcal{K}}) > \text{gen}(K)$ . Let  $\mathcal{H}_{h+1} := \mathcal{H}_h \cup (\overline{\mathcal{K}} \setminus \{K\})$ . Note that  $\bigcup \overline{\mathcal{K}} = \bigcup \mathcal{H}_{h+1}$ . Thus  $\mathcal{H}_{h+1}$  is a subcollection of  $\mathcal{H}$  such that  $|\mathcal{H}_{h+1}| = |\mathcal{H}_h| + |\overline{\mathcal{K}}| - 1 \leq |\mathcal{H}_h| + 4 \leq (4h + 5) + 4 = 4(h + 1) + 5$ . Since  $\text{gen}(\bigcup \mathcal{H}_{h+1}) = \text{gen}(\bigcup \overline{\mathcal{K}}) > \text{gen}(K) = h + 1$ , it follows that  $\mathcal{H}_{h+1}$  satisfies the required properties.  $\square$

*Remark.* As we mentioned in Section 1, the size bound  $4g + 5$  in Theorem 1 can be improved to  $4g + 4$  if the arrangement of pseudocircles under consideration is strong. To see this, we note that the size bound  $4g + 5$  in the Main Theorem can be improved to  $4g + 4$  if for the base case in the proof we can guarantee the existence of an  $\mathcal{H}_0$  with  $|\mathcal{H}_0| \leq 4$  and such that  $\text{gen}(\mathcal{H}_0) > 0$ . Now if  $\Gamma$  is a strong arrangement of pseudocircles that cannot be embedded into the sphere, [14, Theorem 10] guarantees that there is a subarrangement  $\Gamma_0$  of size at most 4 that cannot be embedded into the sphere. Thus in this case the collection  $\mathcal{H}_0$  of those cycles (in the cluster of graphs  $\mathcal{H}$  associated to  $\Gamma$ ) that correspond to the pseudocircles in  $\Gamma_0$  satisfies  $|\mathcal{H}_0| \leq 4$  and  $\text{gen}(\mathcal{H}_0) > 0$ , as required.

### 3.2 Reducing Theorem 2 to two lemmas

We now show that Theorem 2 is an easy consequence of two lemmas we state below, and whose proofs encompass most of the rest of this paper. These lemmas involve the concept of the degeneracy of a face of an embedded graph, which we now proceed to explain.

First we recall that if  $G$  is an embedded graph, then a *face* of  $G$  is a connected component of  $\mathbb{R}^2 \setminus G$ . If a graph is *cellularly embedded* (that is, if each face is homeomorphic to an open disk), then the collection of facial walks determines the embedding, but this is not true for a non-cellularly embedded graph. In the general case, each face is homeomorphic to a compact surface of some genus  $g \geq 0$  from which a finite number  $m \geq 1$  of points have been removed; here  $g$  is the *genus* of the face, and  $d := m - 1$  its *degeneracy*. For a face with degeneracy  $d$ , there are  $d + 1$  facial walks that bound the face. In a cellular embedding, both the genus and the degeneracy of each face are

equal to zero. Indeed, each face is bounded by a single facial walk (that is, its degeneracy is zero), and each face is homeomorphic to an open disk, that is, to a sphere (compact surface of genus 0) with 1 point removed. If a face has positive degeneracy, then we say it is *degenerate*; otherwise it is *non-degenerate*.

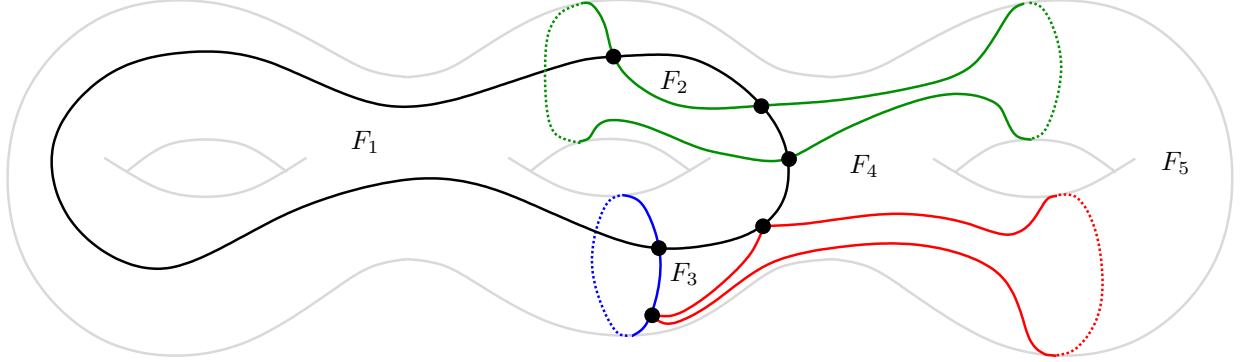


Figure 1: Illustration of the genus and the degeneracy of the faces of an embedded graph. Faces  $F_2, F_3$ , and  $F_4$  have both genus and degeneracy zero. The fact that  $F_4$  does not have degeneracy zero may not be immediately obvious, but it is readily verified since it does not contain any non-contractible simple closed curve. Face  $F_1$  has genus 1 and degeneracy zero, and  $F_5$  has genus zero and degeneracy 1.

We illustrate the concepts of genus and degeneracy of a face in Figure 1. In this embedded graph, each of the faces  $F_2, F_3$ , and  $F_4$  is homeomorphic to an open disk (that is, to a sphere minus one point), and so it has both genus and degeneracy zero. Face  $F_1$  is homeomorphic to a torus minus one point, so it has genus 1 and degeneracy zero. Finally,  $F_5$  is homeomorphic to a sphere minus two points (note that it is bounded by two facial walks), and so it has genus zero and degeneracy 1.

We are ready to state the lemmas that, put together, imply Theorem 2. We note that in both lemmas we assume that  $\bigcup \mathcal{H}$  is cellular. This is an essential assumption for the proofs of these lemmas but, as we shall see shortly, Theorem 2 will follow even if its statement does not include this as a hypothesis.

The first key lemma is the following, which we informally capture by saying that “if the anchor has a degenerate face, then at most 2 graphs of the cluster need to be added to the anchor, to obtain a graph whose genus is greater than the genus of the anchor”.

**Lemma 3.** *Let  $\mathcal{H}$  be a cluster of graphs with anchor  $H$ , such that  $\bigcup \mathcal{H}$  is cellular. Suppose that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ , and that  $H$  has a degenerate face. Then there is a collection  $\mathcal{H}' \subseteq \mathcal{H}$ , that includes  $H$  and satisfies  $|\mathcal{H}'| \leq 3$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H}')$ .*

We now state the second key lemma. Informally speaking, this says that “if all the faces of the anchor are non-degenerate, then at most two graphs of the cluster need to be added to the anchor, so that the resulting graph either (i) has greater genus than the anchor; or (ii) has a degenerate face”. Formally:

**Lemma 4.** *Let  $\mathcal{H}$  be a cluster of graphs with anchor  $H$ , such that  $\bigcup \mathcal{H}$  is cellular. Suppose that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ , and that every face of  $H$  is non-degenerate. Then there is a collection  $\mathcal{H}'' \subseteq \mathcal{H}$ , that includes  $H$  and satisfies  $|\mathcal{H}''| \leq 3$ , such that either (i)  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H}'')$ ; or (ii)  $\bigcup \mathcal{H}''$  has a degenerate face.*

Most of the rest of the paper is devoted to proving these lemmas. We close this section by showing that they imply Theorem 2.

*Proof of Theorem 2, assuming Lemmas 3 and 4.* First we show that if Theorem 2 holds when  $\bigcup \mathcal{H}$  is cellular, then it always holds. Suppose that  $\mathcal{I}$  is a cluster of graphs with anchor  $I$ , where  $\text{gen}(I) < \text{gen}(\bigcup \mathcal{I})$ , and  $\bigcup \mathcal{I}$  is not cellular. Every embedded graph is isomorphic to a cellularly embedded graph, and in particular there exists a cluster of graphs  $\mathcal{H}$  such that  $\bigcup \mathcal{H}$  is cellular, and an isomorphism  $\phi : \bigcup \mathcal{I} \rightarrow \bigcup \mathcal{H}$  that maps each element of  $\mathcal{I}$  to an element of  $\mathcal{H}$ . The image  $H$  of  $I$  under  $\phi$  is then an anchor of  $\mathcal{H}$ , and  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ .

Suppose that Theorem 2 holds for  $\mathcal{H}$ . Thus there is an  $\overline{\mathcal{H}} \subseteq \mathcal{H}$ , with  $H \in \overline{\mathcal{H}}$  and  $|\overline{\mathcal{H}}| \leq 5$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \overline{\mathcal{H}})$ . Then the collection  $\overline{\mathcal{I}} := \{\phi^{-1}(K) \mid K \in \overline{\mathcal{H}}\}$  contains  $I$ , satisfies  $|\overline{\mathcal{I}}| \leq 5$ , and  $\text{gen}(I) < \text{gen}(\bigcup \overline{\mathcal{I}})$ . That is, Theorem 2 also holds for  $\mathcal{I}$ . Therefore, as claimed, it suffices to prove the theorem under the assumption that  $\bigcup \mathcal{H}$  is cellular.

Thus we let  $\mathcal{H} = \{H, H_1, \dots, H_n\}$  be a cluster of graphs with anchor  $H$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ , and  $\bigcup \mathcal{H}$  is cellular. If  $H$  has a degenerate face, then Theorem 2 follows immediately from Lemma 3. Thus we suppose that all the faces of  $H$  are non-degenerate, and apply Lemma 4. Thus there exist (not necessarily distinct) graphs  $H_i, H_j \in \mathcal{H} \setminus \{H\}$  such that either (i)  $\text{gen}(H \cup H_i \cup H_j) > \text{gen}(H)$ ; or (ii)  $H \cup H_i \cup H_j$  has a degenerate face. In the first case we are done by letting  $\overline{\mathcal{H}} := \{H, H_i, H_j\}$ . Thus we assume that (ii) holds, and (i) does not, that is,  $\text{gen}(H \cup H_i \cup H_j) = \text{gen}(H)$ .

Since  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ , it then follows that  $\text{gen}(H \cup H_i \cup H_j) < \text{gen}(\bigcup \mathcal{H})$ , and so  $\mathcal{J} := \{H_r \in \mathcal{H} \mid r \notin \{i, j\}\}$  is not empty. The collection  $\mathcal{K} := \{H \cup H_i \cup H_j\} \cup \mathcal{J}$  is then a cluster of graphs with anchor  $H \cup H_i \cup H_j$ , since the anchor property of  $H$  in  $\mathcal{H}$  is obviously inherited to  $H \cup H_i \cup H_j$  in  $\mathcal{K}$ .

Since  $\bigcup \mathcal{K} = \bigcup \mathcal{H}$ , it follows that  $\text{gen}(H \cup H_i \cup H_j) < \text{gen}(\bigcup \mathcal{K})$ . Recall that the anchor  $H \cup H_i \cup H_j$  of  $\mathcal{K}$  has a degenerate face. Thus we can apply Lemma 3 to  $\mathcal{K}$ , to obtain that there exist (not necessarily different) graphs  $H_k, H_\ell$  in  $\mathcal{J}$  such that  $\text{gen}((H \cup H_i \cup H_j) \cup H_k \cup H_\ell) > \text{gen}(H \cup H_i \cup H_j)$ . Therefore we have  $\text{gen}(H \cup H_i \cup H_j \cup H_k \cup H_\ell) > \text{gen}(H)$ , and so we are done by setting  $\overline{\mathcal{H}} := \{H, H_i, H_j, H_k, H_\ell\}$ .  $\square$

## 4 Proof of Lemma 3

Let  $\mathcal{H} = \{H, H_1, \dots, H_n\}$  be a cluster of graphs such that  $\bigcup \mathcal{H}$  is cellular,  $H$  is an anchor of  $\mathcal{H}$ , and  $H$  has a degenerate face  $F$ . Let  $\mathcal{W}$  be the set of facial walks of  $H$  that bound  $F$ . The degeneracy of  $F$  means that  $|\mathcal{W}| \geq 2$ . We refer to reader to Figure 2(a), where we illustrate an anchor  $H$  in the double torus, and a face  $F$  that is homeomorphic to a disk minus two points; thus the degeneracy of the face  $F$  in this example is exactly 1.

We let  $I$  denote the subgraph of  $\bigcup \mathcal{H}$  induced by the edges contained in the face  $F$ . Since  $\bigcup \mathcal{H}$  is cellular and  $F$  is a degenerate face of  $H$  (in particular,  $F$  is not homeomorphic to an open disk), it follows that  $I$  is not a null graph, that is,  $I$  has at least one edge.



To help comprehension, we say that the edges of  $H_i$  are of colour  $i$ , for  $i = 1, \dots, n$ . Thus every edge of  $\mathcal{H} \setminus \{H\}$  has (exactly) one colour; in particular, every edge in  $I$  is in  $\mathcal{H} \setminus \{H\}$ , and so it has one colour. A subgraph of  $\mathcal{H}$  is *monochromatic* if all its edges are of the same colour.

The graph  $I$  can be decomposed as the edge-disjoint union of graphs  $G_1, G_2, \dots, G_m$  such that, for  $k = 1, \dots, m$ ,  $G_k$  is a connected monochromatic subgraph of  $I$ , and it is maximal with respect to these properties. Note that it may be that  $m > n$ ; indeed, even though each element of  $\mathcal{H}$  is connected, the intersection of  $H_i$  with  $F$  may be disconnected for some  $i \in \{1, 2, \dots, n\}$ .

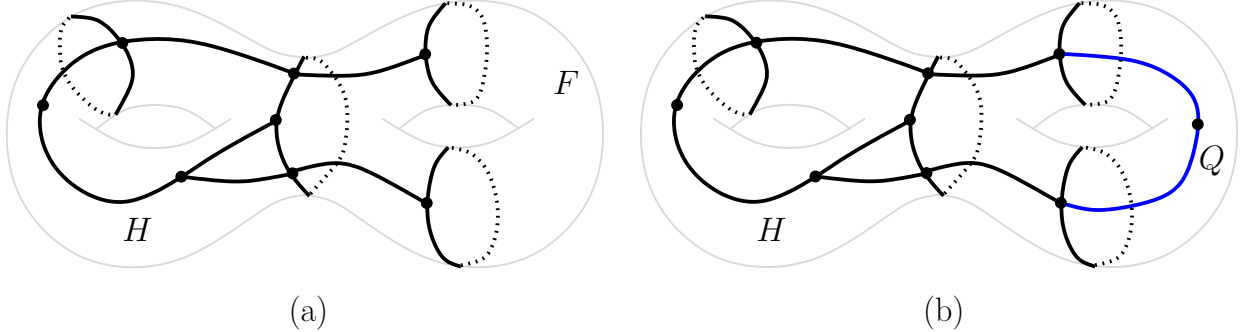


Figure 2: In (a) we depict the anchor  $H$  of a cluster of graphs  $\mathcal{H}$  (the other graphs of  $\mathcal{H}$  are not shown). Here  $H$  has a face  $F$  bounded by two facial walks. Thus  $F$  has degeneracy 1. In (b) we illustrate a path  $Q$  contained in  $F$ , except for its endpoints, one of which lies on  $W_1$ , and the other one lies on  $W_2$ .

The connectedness of each  $H_i \in \mathcal{H}$  implies that  $G_k$  has at least one vertex in common with some walk in  $\mathcal{W}$ , for  $k = 1, \dots, m$ . Indeed, suppose that some  $G_k \in \{G_1, \dots, G_m\}$  has no vertex in common with any walk in  $\mathcal{W}$ ; thus  $G_k$  is completely (including its vertices) contained in  $F$ . Let  $i$  be the colour of the edges in  $G_k$ . Since  $H_i$  is connected, it follows that  $H_i$  must equal  $G_k$ , and in particular, that  $H_i$  does not intersect  $H$ . But this is impossible, since  $H$  is an anchor of  $\mathcal{H}$ . From this observation it follows that  $\mathcal{G} := \{H, G_1, \dots, G_m\}$  is a cluster of graphs with anchor  $H$ .

We claim that to prove the lemma it is enough to show that there exists a subcollection  $\mathcal{G}'$  of  $\mathcal{G}$ , that includes  $H$  and satisfies  $|\mathcal{G}'| \leq 3$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{G}')$ . For suppose such a  $\mathcal{G}'$  exists, then  $\mathcal{G}' = \{H, G_k, G_\ell\}$  for some (non-necessarily distinct)  $k, \ell \in \{1, \dots, m\}$ . Now let  $i$  (respectively,  $j$ ) be the colour of the edges in  $G_k$  (respectively,  $G_\ell$ ). Thus  $G_k$  is a subgraph of  $H_i$ , and  $G_\ell$  is a subgraph of  $H_j$ . Let  $\mathcal{H}' := \{H, H_i, H_j\}$ . Since  $G_k \cup G_\ell \subseteq H_i \cup H_j$ , then  $\text{gen}(\bigcup \mathcal{G}') \leq \text{gen}(\bigcup \mathcal{H}')$ , and so  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{G}')$  implies that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H}')$ . Thus the lemma follows, since  $\mathcal{H}'$  satisfies the required properties.

Thus we devote the rest of the proof to show that there exists a subcollection  $\mathcal{G}'$  of  $\mathcal{G}$ , that includes  $H$  and satisfies  $|\mathcal{G}'| \leq 3$ , such that  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{G}')$ .

Let  $G_i \in \mathcal{G}$ . If for a walk  $W \in \mathcal{W}$  the graph  $G_i$  has a vertex in common with  $W$ , we say that  $G_i$  *attaches to*  $W$ . We recall that each  $G_i \in \mathcal{G}$  attaches to at least one walk in  $\mathcal{W}$ .

We first deal with the case in which there is a  $G_k \in \mathcal{G}$  that attaches to two distinct facial walks  $W_1, W_2 \in \mathcal{W}$ . In this case there is a path  $P$  from a vertex  $u \in W_1$  to a vertex  $v \in W_2$  that is contained in  $F$  except for its endpoints, and such that  $P$  is contained in  $G_k$ . We claim that  $\text{gen}(H) < \text{gen}(H \cup P)$ . Note that this settles the lemma in this case by setting  $\mathcal{G}' = \{H, G_k\}$ , as the fact that  $P$  is contained in  $G_k$  implies that  $\text{gen}(H \cup P) \leq \text{gen}(H \cup G_k)$ .

To see that  $\text{gen}(H) < \text{gen}(H \cup P)$ , we note that if  $q$  is the length (number of edges) of  $P$ , then  $P \cup H$  has  $q$  more edges,  $q - 1$  more vertices, and one fewer facial walk than  $H$  ( $P$  collapses  $W_1$  and  $W_2$  into a single facial walk). Thus an elementary counting gives that  $\text{gen}(P \cup H) = \text{gen}(H) + 1$ .

In the remaining case, each  $G_i \in \mathcal{G}$  attaches to exactly one facial walk in  $\mathcal{W}$ . Let  $W_1, W_2, \dots, W_r$  be the elements of  $\mathcal{W}$ . Thus if we say that  $G_i \in \mathcal{G}$  is of *type*  $s$  if it attaches to walk  $W_s$ , then each  $G_i \in \mathcal{G}$  is of type  $s$  for exactly one  $s \in \{1, 2, \dots, r\}$ . We claim that there is a  $G_i \in \mathcal{G}$  of type 1 that intersects a graph of type  $s$  for some  $s \geq 2$ . Seeking a contradiction, suppose that this is not the case. Then there exists a simple closed curve  $\alpha$  contained in  $F$ , with the following properties:

- (a)  $\alpha$  does not intersect  $G_1 \cup G_2 \cup \dots \cup G_m$  (and therefore does not intersect  $\bigcup \mathcal{H}$ ); and
- (b)  $F \setminus \alpha$  has two components, one that contains all the edges of type 1 and one that contains all the edges of type  $s$  for  $s \geq 2$ .

Now  $\alpha$  is a non-separating curve in the host surface of  $\mathcal{H}$ , and since it does not intersect  $\bigcup \mathcal{H}$  it follows that the face of  $\bigcup \mathcal{H}$  that contains  $\alpha$  is not homeomorphic to an open disk, contradicting the cellularity of  $\bigcup \mathcal{H}$ .

Hence there exist a  $G_k \in \mathcal{G}$  of type 1, and a  $G_\ell \in \mathcal{G}$  of type  $s \geq 2$ , such that  $G_k$  and  $G_\ell$  have at least one common vertex. It follows that there is a path  $Q$  contained in  $G_k \cup G_\ell$ , with one endpoint in  $W_1$  and the other endpoint in  $W_s$ , and that except for these endpoints is contained in  $F$ . Let  $r$  denote the length of  $Q$ . Then  $H \cup Q$  has  $r$  more edges,  $r - 1$  more vertices, and one fewer facial walk than  $H$  (here  $Q$  collapses  $W_1$  and  $W_2$  into a single facial walk). Again an elementary counting yields that  $\text{gen}(H) < \text{gen}(H \cup Q)$ , and so  $\text{gen}(H) < \text{gen}(H \cup G_k \cup G_\ell)$ . Thus in this case we are done by setting  $\mathcal{G}' := \{H, G_k, G_\ell\}$ .  $\square$

## 5 Proof of Lemma 4

The proof of Lemma 4 consists of two steps, which are stated as Claims A and B below. For the proof of Claim B we assume the following statement on non-separating cycles in clusters of graphs whose anchor consists of a single vertex. We recall that if  $D$  is a cycle in a graph embedded in a surface  $\Sigma$  such that  $\Sigma \setminus D$  is connected, then  $D$  is *non-separating*.

**Proposition 5.** *Let  $\mathcal{G}$  be a cluster of graphs, where the anchor  $G$  consists of a single vertex, and  $\bigcup \mathcal{G}$  is cellularly embedded in  $\Sigma_g$  for some  $g \geq 1$ . Then there is a subcollection  $\mathcal{G}_\circ \subseteq (\mathcal{G} \setminus \{G\})$ , with  $|\mathcal{G}_\circ| \leq 2$ , such that  $\bigcup \mathcal{G}_\circ$  contains a non-separating cycle.*

In this section we show that, assuming Proposition 5, Lemma 4 follows. The proof of Proposition 5 is deferred to Sections 6 and 7.

Throughout this section,  $\mathcal{H} = \{H, H_1, \dots, H_n\}$  is a cluster of graphs with anchor  $H$  as in the statement of Lemma 4. Thus  $\bigcup \mathcal{H}$  is cellular,  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$ , and every face of  $H$  is non-degenerate. We let  $\Sigma_g$  be the host surface of  $\mathcal{H}$ .

The assumption  $\text{gen}(H) < \text{gen}(\bigcup \mathcal{H})$  implies that  $H$  is not cellularly embedded in  $\Sigma_g$ . Now any graph that is not cellularly embedded has a face that either is degenerate or has positive genus. Since by assumption every face of  $H$  is non-degenerate, it follows that  $H$  has a face  $F$  with positive genus (and degeneracy zero). That is,  $F$  is a homeomorphic to a compact surface of positive genus from which a single point has been removed. Let  $W$  be the unique facial walk of  $H$  that bounds  $F$ .

The situation is illustrated in (a) of Figure 3. The anchor  $H$  is contained in the left handle of the double torus, and it is easy to see that  $\text{gen}(H) = 1$  (that is,  $H$  “fills the left handle”). In this case  $F$  has genus 1, as it is homeomorphic to a torus minus one point.

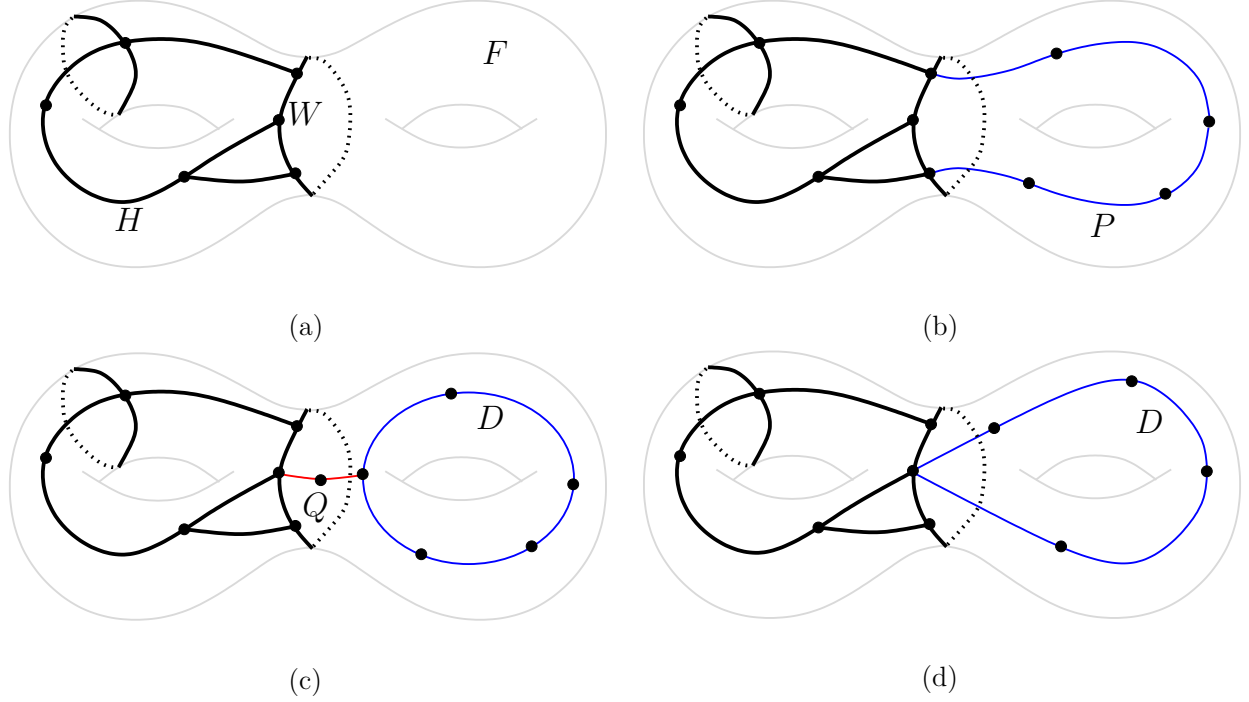


Figure 3: In (a) we depict the anchor  $H$  of a cluster of graphs  $\mathcal{H}$  (the other graphs of  $\mathcal{H}$  are not shown), where  $H$  has a face  $F$  with degeneracy zero and genus 1. The (unique) facial walk of  $H$  that bounds  $F$  is  $W$ . In (b), (c), and (d) we show the three structures involved in Claim A. In (b) we show an  $F$ -non-separating path. In (c) we have a non-separating cycle  $D$  completely contained in  $F$ ; in particular,  $D$  is disjoint from  $W$ . (The path  $Q$  also shown in (c), with one endpoint in  $W$  and one endpoint in  $D$ , is used in the proof of Claim A). Finally, in (d) we show a non-separating cycle  $D$  that is contained in  $F$ , except for a single vertex that lies on  $W$ .

Now suppose that  $P$  is a path that is contained in  $F$  except for its endpoints  $u$  and  $v$ , which lie on  $W$ . We say that such a path is  *$F$ -non-separating* if there is a path  $R$  from  $u$  to  $v$ , contained in  $W$ , such that  $R \cup P$  is a non-separating cycle. An  $F$ -non-separating path is illustrated in (b) of Figure 3.

Lemma 4 is an immediate consequence of the following two claims, whose proofs encompass the rest of this section.

**Claim A.** *Suppose that there exists a subcollection  $\mathcal{H}_0$  of  $\mathcal{H}$ , with  $|\mathcal{H}_0| \leq 2$ , such that  $\bigcup \mathcal{H}_0$  contains one of the following:*

- (i) *An  $F$ -non-separating path.*
- (ii) *A non-separating cycle contained in  $F$ , except perhaps for a single vertex that lies on  $W$ .*

Then Lemma 4 holds by setting  $\mathcal{H}'' = \{H\} \cup \mathcal{H}_\circ$ .

Possibility (i) in the statement of Claim A is illustrated in (b) of Figure 3. The two possible situations in (ii) (a non-separating cycle completely contained in  $F$ , and a non-separating cycle contained in  $F$  except for a single vertex that lies on  $W$ ) are illustrated in (c) and (d) of Figure 3, respectively. (The path  $Q$  in (d) is used in the proof of Claim A).

**Claim B.** *There exists a subcollection  $\mathcal{H}_\circ$  of  $\mathcal{H}$  with the properties stated in Claim A.*

## 5.1 Proof of Claim A

*Proof.* Consider the subcollection  $\mathcal{H}_\circ$  whose existence is assumed in Claim A. Thus  $\mathcal{H}_\circ$  consists either of one or two elements in  $\mathcal{H}$ . To simplify the discussion, it is valid to say that  $\mathcal{H}_\circ = \{H_k, H_\ell\}$ , where  $H_k$  and  $H_\ell$  are *not necessarily distinct* elements of  $\mathcal{H}$ .

Our first step is to produce, from one of the structures in the statement of Claim A (an  $F$ -non-separating path or a non-separating cycle) a subgraph  $L$  of  $H_k \cup H_\ell$  with certain properties. If there is an  $F$ -non-separating path  $P$ , then we let  $L := P$ . If there is a non-separating cycle  $D$  contained in  $F$ , except for a single vertex that lies on  $W$ , then we let  $L := D$ . In the alternative, there is a non-separating cycle  $D$  completely contained in  $F$ . In this case, since  $H_k$  and  $H_\ell$  are connected, and each of them has at least one vertex in common with  $H$ , it follows that there is a path  $Q$  contained in  $H_k \cup H_\ell$ , with one endpoint in  $D$  and the other endpoint in  $W$ , that is otherwise disjoint from  $D \cup W$ . (See Figure 3(c)). Thus  $Q$  is contained in  $F$ , except for its endpoint in  $W$ . In this case we let  $L := D \cup Q$ .

We note that in every case  $L$  satisfies the following properties:

1.  $L$  is a subgraph of  $H_k \cup H_\ell$ .
2.  $L$  is contained in  $F$ , except for some vertices (either one or two) that lie on  $W$ .
3.  $H \cup L$  has a degenerate face. (This is the important structural property).

Properties 1 and 2 follow immediately from the construction of  $L$ . To see that Property 3 holds, note that  $F \setminus L$  is a face  $F_L$  of  $H \cup L$ . It is immediately verified that (regardless of whether  $L$  is a path, or a cycle, or a cycle plus a path)  $F_L$  bounds exactly two facial walks of  $H \cup L$ , and thus it is degenerate.

The argument to finish the proof of Claim A is heavily based on the following remark. In what follows, if  $e$  is an edge of a connected graph  $G$  that is incident with a degree 1 vertex, then we use  $G - e$  to denote the graph that results by removing from  $G$  both  $e$  and this degree 1 vertex. If  $e$  is not incident with any degree 1 vertex, then  $G - e$  is simply the graph that results by removing  $e$  from  $G$ .

**Observation.** *Let  $G$  be a connected embedded graph, and let  $e$  be an edge of  $G$  such that  $G - e$  is also connected. If  $G - e$  has a degenerate face, then either (a)  $\text{gen}(G) > \text{gen}(G - e)$ ; or (b)  $G$  has a degenerate face.*

*Proof of the Observation.* Let  $G$  and  $e$  be as in the statement of the Observation, and let  $J$  be a degenerate face of  $G - e$ . If  $e$  is not inside the face  $J$ , then  $J$  is also a face of  $G$ , with the same facial walks as in  $G - e$ ; thus in this case  $G$  has a degenerate face.

In the alternative,  $e$  is inside  $J$ . In this case  $J$  (which is a face of  $G - e$ ) is not a face of  $G$ , but is contained in a face of  $G$ . Now  $e$  must have at least one endpoint  $u$  in a facial walk  $U$  of  $J$ , and the other endpoint  $v$  of  $e$  is either inside  $J$  or lies on a facial walk of  $J$ . If  $v$  is inside  $J$  then it is a degree one vertex of  $G$ , and  $J - e$  is a face of  $G$  with the same number of facial walks as  $J$ ; in particular, in this case  $G$  has a degenerate face, namely  $J - e$ . Thus it only remains to analyse the case in which  $v$  lies on a facial walk  $U'$  of  $J$ .

If  $U$  and  $U'$  are distinct facial walks, then  $G$  has one more edge and one fewer facial walk than  $G - e$  (the facial walks  $U, U'$  get collapsed into a single facial walk by the addition of  $e$ ). Thus in this case an elementary counting argument shows that  $\text{gen}(G) = \text{gen}(G - e) + 1 > \text{gen}(G - e)$ , and so we are done.

Thus we are left with the case in which  $U$  and  $U'$  are the same facial walk. We note that then  $U$  together with  $e$  induces two facial walks  $U_1, U_2$  in  $G$ .

Assume first that  $J - e$  is connected. In this case  $J - e$  is a face of  $G$ , which is bounded by (at least) the facial walks  $U_1$  and  $U_2$ ; in particular,  $G$  has a degenerate face, and so we are done. Finally, suppose that  $J - e$  is disconnected. Since  $J$  is connected (it is a face of  $G - e$ ) it follows that  $J - e$  has exactly two components  $J_1, J_2$ , which are faces of  $G$ . One of these faces bounds  $U_1$ , and the other bounds  $U_2$ . Without loss of generality,  $J_1$  bounds  $U_1$  and  $J_2$  bounds  $U_2$ . Now since  $J$  is degenerate, it follows that there is a facial walk  $U''$ , distinct from  $U$ , that also bounds  $J$ . Then  $U''$  must be bounded by either  $J_1$  or  $J_2$ . In the former case,  $J_1$  is degenerate, and in the latter case  $J_2$  is degenerate.  $\square$

We are finally ready to finish the proof of Claim A, by showing that Lemma 4 follows by letting  $\mathcal{H}'' := \mathcal{H}_\circ = \{H, H_k, H_\ell\}$ . That is, we will show that either  $\text{gen}(H) < \text{gen}(H \cup H_k \cup H_\ell)$ , or  $H \cup H_k \cup H_\ell$  has a degenerate face.

The connectedness of  $H_k$  and  $H_\ell$ , and the fact that each of these graphs has at least one vertex in common with  $H$ , implies that there is a sequence  $L_0, L_1, \dots, L_m$  of subgraphs of  $H \cup H_k \cup H_\ell$  such that the following hold: (i)  $L_0 = H \cup L$ ; (ii)  $L_m = H \cup H_k \cup H_\ell$ ; (iii) for  $i = 0, 1, \dots, m-1$  there is an edge  $e_i$  of  $H_k \cup H_\ell$  such that  $L_i = L_{i+1} - e_i$ . Roughly speaking, starting from  $H \cup H_k \cup H_\ell$  we can obtain  $H \cup L$  by successively removing edges (if an edge is incident with a degree one vertex, we also remove that vertex, as we remarked before the Observation above), so that at every step we have a connected graph.

If  $L_m = H \cup H_k \cup H_\ell$  has a degenerate face then we are done (as then (ii) holds in the statement of Lemma 4). Thus we assume that  $L_m$  has no degenerate face. Let  $j$  be the smallest integer in  $\{0, 1, \dots, m\}$  such that  $L_j$  has no degenerate face. By assumption  $L_0$  has a degenerate face, so  $j \geq 1$ . Thus  $L_{j-1}$  does have a degenerate face, and since  $L_{j-1} = L_j - e_{j-1}$ , we can apply the Observation above, obtaining that  $\text{gen}(L_j) > \text{gen}(L_{j-1})$ . Since  $H \subseteq H \cup L \subseteq L_{j-1}$  and  $L_j \subseteq H \cup H_k \cup H_\ell$ , it follows that  $\text{gen}(H) \leq \text{gen}(L_{j-1}) < \text{gen}(L_j) \leq \text{gen}(H \cup H_k \cup H_\ell)$ , and so (i) in the statement of Lemma 4 holds.  $\square$

## 5.2 Proof of Claim B

*Proof.* Since  $\bigcup \mathcal{H}$  is cellularly embedded, and  $F$  is a face of  $H$  with positive genus, it follows that there exist edges of  $H_1 \cup H_2 \cup \dots \cup H_n$  contained in  $F$ . By relabelling if necessary, we can assume that for some  $m \leq n$ ,  $H_1, H_2, \dots, H_m$  are the graphs in  $\mathcal{H} \setminus \{H\}$  that contain at least one edge in  $F$ .

Now for each  $H_i$  with  $i \in \{1, 2, \dots, m\}$ , let  $I_i$  be the subgraph of  $H_i$  induced by the edges of  $H_i$  inside the face  $F$ . Thus  $\{I_1, \dots, I_m\}$  is a collection whose union is contained in  $F$ , except for the *attachment* vertices, that is, those vertices in the graphs  $I_i$  that are in  $W$  (and thus in  $H$ ). Note that each  $I_i \in \{I_1, \dots, I_m\}$  has at least one vertex of attachment: this follows since each  $H_i \in \mathcal{H}$  is connected, and has at least one vertex in common with the anchor  $H$ : if some  $I_i$  had no vertices of attachment, then it (and hence  $H_i$ ) would not intersect  $H$ . For convenience, we choose to keep the labels of the vertices and edges in  $I_1 \cup I_2 \cup \dots \cup I_m$  as they are inherited from  $H_1 \cup H_2 \cup \dots \cup H_m$ .

Note that even though each  $I_i$  has at least one vertex in common with  $H$ , the collection  $\{H, I_1, I_2, \dots, I_m\}$  may not be a cluster of graphs, since the graphs  $I_i$  are not necessarily connected. However, this is not relevant to our purposes.

We now collapse  $W$  to a point  $u$ , obtaining the compact surface  $\Sigma := F \cup \{u\}$  (we discard  $\Sigma_g \setminus F$ ). Thus  $\Sigma$  has the same genus as  $F$ , and  $I_1 \cup I_2 \cup \dots \cup I_m$  naturally induces an embedded graph  $K$  in  $\Sigma$ : every edge and vertex of  $I_1 \cup I_2 \cup \dots \cup I_m$  is maintained, with the exception of the attachment vertices, which get identified to the single vertex  $u$ . For each  $i = 1, 2, \dots, m$ , we let  $J_i$  be the subgraph of  $K$  naturally induced by  $I_i$ .

By letting  $J$  be the graph that consists solely of vertex  $u$ , the collection  $\mathcal{J} := \{J, J_1, J_2, \dots, J_m\}$  is then a cluster of graphs in  $\Sigma_h$ , as the property that each  $I_i$  had at least one attachment vertex implies that each  $J_i$  contains  $u$ , that is, intersects the anchor  $J$ . With the exception of  $u$ , each vertex or edge of  $J_i$  is inherited from a vertex or edge of  $I_i$  (and hence of  $H_i$ ), and we choose to maintain their respective labels.

The cellularity of  $\bigcup \mathcal{H}$  implies that  $\bigcup \mathcal{J}$  is also cellularly embedded, and so we may apply Proposition 5 to  $\mathcal{J}$ , to obtain that there exists a subcollection  $\mathcal{J}_\circ$  of  $\mathcal{J}$ , with  $|\mathcal{J}_\circ| \leq 2$ , such that  $\bigcup \mathcal{J}_\circ$  contains a non-separating cycle. Thus there exist (not necessarily distinct) integers  $k, \ell$  such that  $J_k \cup J_\ell$  contains a non-separating cycle  $D$ .

To finish the proof it suffices to look at the subgraph  $D'$  of  $\bigcup \mathcal{H}$  (back in  $\Sigma_g$ ) induced by the edges of  $D$ . Note that  $D'$  is contained in  $H_k \cup H_\ell$ . If  $D$  contains  $u$ , then  $D'$  is either an  $F$ -non-separating path or a non-separating cycle that has a single vertex in common with  $W$ . If  $D$  does not contain  $u$ , then  $D'$  is a cycle completely contained in  $F$ . Thus the subcollection  $\mathcal{H}_\circ := \{H_k, H_\ell\}$  of  $\mathcal{H}$  has the required properties.  $\square$

## 6 Towards Proposition 5: short-circuiting non-separating cycles

In the context of the statement of Proposition 5 we have a cluster of graphs  $\mathcal{G} = \{G, G_1, \dots, G_n\}$ , cellularly embedded in some surface  $\Sigma_g$  with  $g \geq 1$ . The graph  $G$  is an anchor of the cluster, and it consists of a single vertex  $a$ . Since  $G$  is edgeless, it follows that each edge  $e$  of  $\bigcup \mathcal{G}$  belongs to  $G_i$  for exactly one  $i \in \{1, \dots, n\}$ . To help comprehension, we say that  $e$  is *of colour  $i$* . Thus each edge of  $\bigcup \mathcal{G}$  has exactly one colour. A subgraph of  $\bigcup \mathcal{G}$  is  *$k$ -coloured* if the number of distinct colours in its edge set is exactly  $k$ .

For brevity, we will refer to a non-separating cycle simply as an *ns-cycle*. Under this terminology, Proposition 5 claims the existence of a 1- or 2-coloured ns-cycle in  $\bigcup \mathcal{G}$ .

The existence of an ns-cycle follows since  $\bigcup \mathcal{G}$  is cellularly embedded in  $\Sigma_g$  for some  $g \geq 1$  (see [14, Lemma 11]). If such a cycle is  $k$ -coloured for some  $k \geq 3$ , we need to find a way to “short-circuit” it to find an ns-cycle with fewer colours. (This short-circuiting idea is also central for the proof of the main theorem in [14]). The proof of Proposition 5 consists of iteratively applying this short-circuiting process, until we obtain a 1- or 2-coloured ns-cycle.

The central idea behind the short-circuiting process is that the set of ns-cycles in an embedded graph satisfies Thomassen's 3-path-condition [16, Proposition 3.5]: if  $R, S, T$  are pairwise internally disjoint paths with the same endpoints, and the cycle  $R \cup S$  is non-separating, then one of the cycles  $R \cup T$  and  $S \cup T$  is non-separating. Thus if we start with an ns-cycle  $R \cup S$  and a suitable path  $T$  internally disjoint from  $R \cup S$  (where both endpoints of  $T$  are in  $R \cup S$ ), we can apply the 3-path-condition to find an ns-cycle with fewer colours than  $R \cup S$ .

In the standard graph theory terminology, a *trail* is a walk in which no edge appears more than once. If the startpoint  $u$  and the endpoint  $v$  of a trail  $T$  are distinct, then  $T$  is a *uv-trail*. A *circuit* is a trail whose startpoint and endpoint are the same. If  $W = v_0 e_1 v_1 \dots e_n v_n$  is a walk on a graph, then  $W^{-1}$  is the *reverse* walk of  $W$ , namely  $v_n e_n v_{n-1} \dots v_1 e_1 v_0$ . If  $W'$  is a walk  $v_n e_{n+1} v_{n+1}, \dots, e_m v_m$  (the endpoint of  $W$  is the startpoint of  $W'$ ), then  $WW'$  is the *concatenation*  $v_0 e_1 v_1 \dots e_n v_n e_{n+1} v_{n+1} \dots e_m v_m$  of  $W$  and  $W'$ .

If  $T$  is not internally disjoint from  $R \cup S$  (or even if  $T$  is not a path, but a trail, and/or  $R \cup S$  is not a cycle but a circuit), we cannot apply the 3-path-condition. Thus we need a version of the 3-path-condition that applies to trails (instead of paths) and circuits (instead of cycles). As we shall see shortly, a property totally analogous to the 3-path-condition holds in the context of trails and circuits, by considering in this more general context (instead of non-separating cycles) the collection of non-null-homologous circuits in an embedded graph. Before moving on to this generalized version of the 3-path-condition, we recall the concepts of a trail and a circuit.

We adopt the (usual) point of view that a circuit is regarded as a cyclic sequence of vertices and edges, so that if  $C = v_0 e_1 v_1 \dots e_n v_0$  is a circuit, then  $C$  is identical to the circuit  $v_i e_{i+1} v_{i+1} \dots e_{i-1} v_i$ , for all  $i = 1, \dots, n-1$ . If  $C = v_0 e_1 v_1 \dots e_i v_i e_{i+1} \dots e_n v_0$  is a circuit such that  $v_0 = v_i$  for some  $i \neq 0$ , then  $v_0 e_1 v_1 \dots v_i = v_0$  is a *subcircuit* of  $C$ .

We extend the notion of an ns-cycle to circuits, by means of simplicial homology over  $\mathbb{Z}_2$ . From this viewpoint, a cycle is an ns-cycle if (and only if) it is non-null-homologous. Thus we say that a circuit is an *ns-circuit* if it is non-null-homologous. The following trivial observation from elementary homology theory will be repeatedly invoked in the short-circuiting iterative process in the proof of Proposition 5.

*Remark.* Every ns-circuit contains an ns-cycle as a subcircuit.

We are now ready to state the extension (to trails and circuits) of the fact that the set of ns-cycles in an embedded graph satisfies Thomassen's 3-path-condition.

**Observation 6** (3-trail condition for ns-circuits). *Let  $T_1, T_2, T_3$  be edge-disjoint trails in an embedded graph, with the same startpoint and the same endpoint. If  $T_1 T_2^{-1}$  is an ns-circuit, then at least one of  $T_1 T_3^{-1}$  and  $T_3 T_2^{-1}$  is also an ns-circuit.*

Some variants of this observation are usually stated without proof (see for instance [4, Section 3.1]), as it is a trivial exercise in homology theory. We give the proof for completeness.

*Proof.* Regarding the circuits as 1-chains, we have that  $T_1 T_3^{-1} + T_3 T_2^{-1} = T_1 T_2^{-1}$ , since  $T_3$  and  $T_3^{-1}$  cancel each other. Since by assumption  $T_1 T_2^{-1}$  is non-null-homologous, it follows that at least one of  $T_1 T_3^{-1}$  and  $T_3 T_2^{-1}$  must also be non-null-homologous.  $\square$

With Observation 6 in our toolkit, we are finally ready to prove Proposition 5.

## 7 Proof of Proposition 5

*Proof.* Let  $G_1, \dots, G_n$  be the elements in  $\mathcal{G} \setminus \{G\}$ . As we mentioned in the previous section, to help comprehension we say that the edges of  $G_i$  are of colour  $i$ , for  $i = 1, 2, \dots, n$ . Since  $G$  consists of a single vertex  $a$ , and  $G_1, G_2, \dots, G_n$  are pairwise edge-disjoint, it follows that each edge of  $\bigcup \mathcal{G}$  has exactly one colour. If  $T$  is a trail in  $\bigcup \mathcal{G}$  with all the edges of the same colour  $i$  for some  $i \in \{1, 2, \dots, n\}$ , then  $T$  is *monochromatic*, and we say that  $i$  is the colour of  $T$ .

If  $C$  is a circuit in  $\bigcup \mathcal{G}$ , then  $C$  can be written as a concatenation  $T_0 T_1 \dots T_{r-1}$  of maximal monochromatic trails. That is, for  $i = 0, 1, \dots, r-1$ ,  $T_i$  is monochromatic, and the colour of  $T_i$  is distinct from the colour of  $T_{i+1}$  (indices are read modulo  $r$ , and so  $T_{r-1}$  and  $T_0$  are of different colours). We remark that  $T_i$  and  $T_j$  may be of the same colour for some  $i \neq j$ , as long as  $j \notin \{i-1, i+1\}$ . This decomposition of  $C$  as a concatenation of maximal monochromatic trails is unique, up to a cyclic permutation of the trails. This uniqueness allows us to call  $T_0 T_1 \dots T_{r-1}$  the *canonical decomposition* of  $C$ ; we call  $r$  the *rank* of the circuit  $C$ .

To prove Proposition 5 we show that there exists an ns-cycle of rank at most 2 (see Statement (4) below), and therefore a subset of  $\{G_1, \dots, G_n\}$  of size at most 2, whose union contains an ns-cycle, as required in the statement of the proposition.

Thus the final goal is to prove Statement (4) below. To help comprehension, we break the proof into several steps. As we will see, showing the existence of an ns-cycle of rank at most 3 is fairly easy (see Statement (2) below). Most of the work is involved with bringing the rank down to at most 2.

(1) *If there exists an ns-circuit of rank  $r$ , then there exists an ns-cycle of rank at most  $r$ .*

*Proof.* This follows immediately from the definition of an ns-circuit. Indeed, if  $C$  is an ns-circuit of rank  $r$ , then it contains an ns-cycle  $D$  as a subcircuit (see Remark before Observation 6); it is readily checked that the rank of  $D$  is at most  $r$ . It is worth noting that the property that  $D$  is a subcircuit of  $C$  (and not just an arbitrary ns-cycle contained in  $C$ ) is essential in order to guarantee that the rank of  $D$  is at most the rank of  $C$ .  $\square$

(2) *There exists an ns-cycle of rank at most 3.*

*Proof.* The existence of an ns-cycle in  $\bigcup \mathcal{G}$  follows from [14, Lemma 11], since  $\bigcup \mathcal{G}$  is cellularly embedded in a surface of positive genus. In order to prove (2) it suffices to show that if  $D$  is an ns-cycle with canonical decomposition  $P_0 P_1 \dots P_{r-1}$ , where  $r \geq 4$ , then there exists an ns-cycle whose rank is smaller than  $r$ ; an iterative application of this fact, starting with an arbitrary ns-cycle, yields the existence of an ns-cycle with rank at most 3.

Suppose first that there exists an  $i \in \{1, \dots, n\}$  such that there are at least two paths in  $\{P_0, P_1, \dots, P_{r-1}\}$  that are of colour  $i$  (note that since  $D$  is a cycle, the elements in its canonical decomposition  $P_0 P_1 \dots P_{r-1}$  are paths). Since  $G_i$  is connected, it follows that there exist distinct  $P_j, P_k$  in the decomposition, both of colour  $i$ , and a path  $R$  of colour  $i$  whose startpoint  $u$  is in  $P_j$  and whose endpoint  $v$  is in  $P_k$ , and such that  $R$  does not contain any edge of  $D$ . Now let  $P, Q$  be the two  $uv$ -paths contained in  $D$  (thus  $D = PQ^{-1}$ ). Then  $P, Q$ , and  $R$  are pairwise edge-disjoint  $uv$ -paths. It is readily verified that since  $R$  is of colour  $i$ , then the rank of each of  $PR^{-1}$  and  $RQ^{-1}$  is strictly smaller than  $r$ . Now since  $D = PQ^{-1}$  is an ns-cycle, and in particular an ns-circuit, it follows from Observation 6 that one of  $PR^{-1}$  and  $RQ^{-1}$  is also an ns-circuit. Thus, one of these is an ns-circuit (and by (1), an ns-cycle) whose rank is smaller than  $r$ .



Suppose finally that all the paths  $P_0, P_1, \dots, P_{r-1}$  are of distinct colours. By relabelling if necessary, we may assume that  $P_0$  is of colour 0, and  $P_2$  is of colour 2. Since  $G_0$  and  $G_2$  are connected and have at least one vertex in common (namely the vertex  $a$  in the anchor  $G$ ), it follows that there exists a path  $U$  with the following properties:

- (i) one endpoint  $v_0$  of  $U$  is in  $P_0$ , and its other endpoint  $v_2$  is in  $P_2$ ;
- (ii)  $U$  is the concatenation of a path of colour 0 with a path of colour 2 (one of these two paths may consist of a single vertex); and
- (iii)  $U$  is edge-disjoint from  $D$ .

Now let  $S, T$  be the  $v_0 v_2$ -paths contained in  $D$ . It is easily seen that the rank of both  $SU^{-1}$  and  $UT^{-1}$  is smaller than  $r$ . Now since  $D = ST^{-1}$  is an ns-cycle (and in particular an ns-circuit), it follows from Observation 6 that one of  $SU^{-1}$  and  $UT^{-1}$  is also an ns-circuit. Thus there exists an ns-circuit (and by (1), an ns-cycle) whose rank is smaller than  $r$ .  $\square$

The following statement gets us to the final goal (the existence of an ns-cycle with rank at most 2) in a particular case. Since a reduction to this case appears several times in (4), it is convenient to deal with it before moving on to (4).

- (3) *Let  $C$  be an ns-circuit with canonical decomposition  $T_1 T_2 T_3$ , where the colour of  $T_i$  is  $i$ , for  $i = 1, 2, 3$ . Suppose that  $T_2$  does not contain the startpoint  $v$  of  $T_1$  (which is the endpoint of  $T_3$ ), but there is some edge of colour 2 incident with  $v$ . Then there is an ns-cycle with rank at most 2.*

*Proof.* We start by noting that the connectedness of  $G_2$  and the assumption that there is an edge of colour 2 incident with  $v$ , imply that there is a path  $U$  of colour 2, with startpoint  $v$  and endpoint  $v_2$  in  $T_2$ , which is edge-disjoint from  $C$ . Now let  $S$  be the subtrail of  $C$  obtained by starting at  $v$ , traversing  $T_1$  completely, and then continuing along  $T_2$  until we reach  $v_2$  (it might be that  $v_2$  is the endpoint of  $T_1$ , in which case  $S$  does not contain edges of  $T_2$ , but this is irrelevant). Now let  $T$  be the trail from  $v$  to  $v_2$  such that  $C = ST^{-1}$ ; thus  $T^{-1}$  is obtained by continuing the traversal of  $C$  after we reached  $v_2$ , and in particular  $T_3$  is a subtrail of  $T^{-1}$ .

The circuit  $SU^{-1}$  is the concatenation of  $T_1$  (which has colour 1) with a trail of colour 2, and the circuit  $UT^{-1}$  is the concatenation of a trail of colour 2 with  $T_3$  (which has colour 3). Thus both circuits have rank exactly 2. Since  $ST^{-1} = C$  is an ns-circuit, it follows by Observation 6 that at least one of  $SU^{-1}$  and  $UT^{-1}$  is an ns-circuit. Thus there exists an ns-circuit of rank 2, and by (1) it follows that there is an ns-cycle of rank at most 2.  $\square$

- (4) *There exists an ns-cycle with rank at most 2.*

*Proof.* By (2), there exists an ns-cycle  $D$  with rank at most 3. If the rank of  $D$  is 1 or 2 we are obviously done, so we assume that the rank of  $D$  is exactly 3. Thus  $D$  has a canonical decomposition  $P_1 P_2 P_3$ , where  $P_i$  is a path for  $i = 1, 2, 3$ . By relabelling the subgraphs  $G_1, G_2, \dots, G_n$  if necessary, we may assume that  $P_i$  is in  $G_i$  (that is,  $P_i$  has colour  $i$ ), for  $i = 1, 2, 3$ . We recall that  $G_1, G_2$ , and  $G_3$  have at least one common vertex, namely the vertex  $a$  in the anchor  $G$ .

Suppose first that  $a$  is in  $D$ . By relabelling if necessary, we may assume that  $a$  is in  $P_3$ , and possibly also in  $P_1$  but not in  $P_2$ . If  $a$  is the endpoint of  $P_3$  (and thus the startpoint of  $P_1$ ) then we are done by applying (3) with  $T_i := P_i$  for  $i = 1, 2, 3$ . So we may assume that  $a$  is an internal vertex

of  $P_3$ . Since  $G_1$  is connected, then there is a path  $R$  of colour 1 from  $a$  to a vertex  $v_1$  in  $P_1$ , such that  $R$  has no edges in common with  $D$ . Let  $P, Q$  be the two  $uv_1$ -paths contained in  $D$ , labelled so that every edge of  $P$  is of colour 1 or 3 (hence  $Q$  contains  $P_2$ ). It is easily checked that  $PR^{-1}$  has rank 2, and  $RQ^{-1}$  has rank at most 3. Now  $PQ^{-1} = D$  is an ns-cycle (in particular an ns-circuit), and so by Observation 6 one of  $PR^{-1}$  and  $RQ^{-1}$  is an ns-circuit. If  $PR^{-1}$  is an ns-circuit we are done, since it has rank 2. Thus we may assume that  $RQ^{-1}$  is an ns-circuit, and that its rank is exactly 3. Then  $RQ^{-1}$  is the concatenation of three trails: (i) a trail  $T_1$  of colour 1, which is the concatenation of  $R$  with the subpath of  $P_1$  from  $v_1$  to the endpoint of  $P_1$  (this last subpath may consist of the single vertex  $v_1$ ); (ii) the trail  $T_2 = P_2$ , of colour 2; and (iii) the subpath of  $P_3$  that starts at the startpoint of  $P_3$  and ends at  $a$ ; this last trail is of colour 3, and cannot consist of a single vertex, since  $a$  is an interior vertex of  $P_3$ . Since  $T_1, T_2, T_3$  satisfy the conditions in (3), it follows that there exists an ns-cycle with rank at most 2, as required.

Finally suppose that  $a$  is not in  $D$ . We may assume that no vertex in  $D$  is in  $G_1 \cap G_2 \cap G_3$  (a vertex with this property need not be unique), for if such a vertex exists, we let it play the role of  $a$  and we are done by the discussion above.

Since each of  $G_1, G_2$  and  $G_3$  is connected, it follows that for  $i = 1, 2, 3$  there exists a path  $Q_i$  of colour  $i$  with startpoint  $v_i$  in  $P_i$  and endpoint  $a$ , where  $Q_i$  is edge-disjoint from  $D$ . Note that  $v_1, v_2, v_3$  cannot all be the same vertex, since  $D = P_1P_2P_3$  is a cycle. By relabelling if necessary, we may assume that  $v_1 \neq v_3$ . Let  $U := Q_3Q_1^{-1}$ , and let  $S, T$  be the two paths from  $v_3$  to  $v_1$  contained in  $D$ , where every edge of  $S$  is of colour 1 or 3 (thus  $T$  contains  $P_2$ ). We note that the circuit  $SU^{-1}$  has rank 2 (its canonical decomposition consists of a trail of colour 1 followed by a trail of colour 3), and  $U^{-1}T$  has rank 3 (its canonical decomposition consists of a trail  $T_1$  of colour 1, followed by  $T_2 = P_2^{-1}$  of colour 2, followed by a trail  $T_3$  of colour 3). Since  $ST^{-1} = D$  is an ns-cycle (and thus an ns-circuit) it follows from Observation 6 that at least one of  $SU^{-1}$  and  $UT^{-1}$  is an ns-circuit. In the former case we are done, since  $SU^{-1}$  is then an ns-circuit of rank 2, and by (1) then there exists an ns-cycle of rank at most 2. In the latter case,  $UT^{-1}$  is an ns-circuit of rank 3 whose canonical decomposition  $T_1T_2T_3$  described above satisfies the conditions in (3). Therefore also in this case there exists an ns-circuit (and by (1), an ns-cycle) of rank at most 2.  $\square$

As we observed before (1), Statement (4) completes the proof of the lemma.  $\square$

## 8 Concluding remarks

It is natural to ask if the condition that there is a pseudocircle that intersects all other pseudocircles in the collection is absolutely necessary. To answer this question we note that it is necessary to require some sort of condition along these lines. Indeed, as observed by Ortner in [14, Figure 16], there exist arbitrarily large collections of pseudocircles (whose union is connected) that cannot be embedded into a sphere, and yet the removal of any pseudocircle leaves an arrangement that can be embedded into a sphere.

On the other hand, in order to have some version of Theorem 1 it is not strictly necessary to have a single pseudocircle intersecting all the others; our techniques and arguments are readily adapted under the assumption that there is a subcollection of bounded size that gets intersected by all other pseudocircles. More precisely, if we define an *m-arrangement* of pseudocircles as a collection in which there is a subcollection of size (at most)  $m$  such that every pseudocircle intersects at least one pseudocircle in this subcollection, then it is easy to show that the corresponding version of

Theorem 1 says that an  $m$ -arrangement of pseudocircles is embeddable into  $\Sigma_g$  if and only if all of its subarrangements of size at most  $4g + (2m + 5)$  are embeddable into  $\Sigma_g$ .

We have proved that a strong arrangement of pseudocircles is embeddable into  $\Sigma_g$  if and only if all of its subarrangements of size at most  $4g + 4$  are embeddable into  $\Sigma_g$ . As Ortner showed in [14, Figure 3], there are strong arrangements of size 4 that are not embeddable into a sphere, and yet all its subarrangements are embeddable into a sphere; thus this result cannot be improved for  $g = 0$ . Similarly, for arrangements of pseudocircles (with the more general definition we used throughout this work) the size bound  $4g + 5$  cannot be improved for the case  $g = 0$ . Indeed, the toroidal arrangement shown in Figure 4 has 5 pseudocircles, it cannot be embedded into the sphere, and yet all its subarrangements of size 4 can be embedded into the sphere.

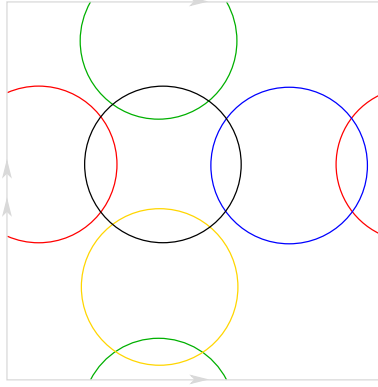


Figure 4: An arrangement of 5 pseudocircles in the torus (given in its polygonal representation). This arrangement is not embeddable into the sphere, but every subarrangement of size 4 is embeddable into the sphere.

Working under the framework of clusters of graphs, one can prove similar results as Theorem 1 to collections of other objects, such as *arcs*, which are homeomorphic images of the interval  $[0, 1]$ . Arrangements of arcs (and, in general, arrangements of curves) are investigated in [5] and [6]. In order to obtain a result along the lines of Theorem 1, some anchorness condition for an arrangement of arcs is required (a discussion analogous to the one given at the beginning of this section applies to these objects as well). If we consider an arrangement of arcs as a collection of arcs that pairwise intersect a finite number of times, and in which there is an arc that intersects all the other arcs in the collection, then the following analogue of Theorem 1 is a consequence of our Main Theorem.

**Theorem 7.** *An arrangement of arcs is embeddable into  $\Sigma_g$  if and only if all of its subarrangements of size at most  $4g + 5$  are embeddable into  $\Sigma_g$ .*

We finally note that in [14], Ortner wrote that one could conjecture that embeddability (of a strong arrangement) into the surface  $\Sigma_g$  of genus  $g$  holds if and only if all  $(4 + g)$ -subarrangements are embeddable into  $\Sigma_g$ . We have proved that, for strong arrangements, embeddability into  $\Sigma_g$  holds if and only if all  $(4 + 4g)$ -subarrangements are embeddable into  $\Sigma$ . The question of whether or not this can be improved to  $4 + g$ , as conjectured in [14], remains open.

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## References

- [1] Pankaj K. Agarwal, Eran Nevo, János Pach, Rom Pinchasi, Micha Sharir, and Shakhar Smorodinsky, *Lenses in arrangements of pseudo-circles and their applications*, J. ACM **51** (2004), no. 2, 139–186.
- [2] J. Bokowski, *Oriented matroids*, in: P.M. Gruber, J.M. Wills (Eds.), *Handbook of convex geometry*, North-Holland, Amsterdam, 1993, pp. 555–602.
- [3] Jürgen Bokowski, Jurij Kovič, Tomaž Pisanski, and Arjana Žitnik, *Combinatorial configurations, quasiline arrangements, and systems of curves on surfaces*. Preprint (2014). [arXiv:1410.2350v1 \[math.CO\]](#).
- [4] Sergio Cabello, Éric Colin de Verdière, and Francis Lazarus, *Finding shortest non-trivial cycles in directed graphs on surfaces*, J. Comput. Geom. **7** (2016), no. 1, 123–148. MR3491090
- [5] Herbert Edelsbrunner, Leonidas Guibas, John Hersherberger, János Pach, Richard Pollack, Raimund Seidel, Micha Sharir, and Jack Snoeyink, *On arrangements of Jordan arcs with three intersections per pair*, Discrete Comput. Geom. **4** (1989), no. 5, 523–539.
- [6] Herbert Edelsbrunner, Leonidas Guibas, János Pach, Richard Pollack, Raimund Seidel, and Micha Sharir, *Arrangements of curves in the plane—topology, combinatorics, and algorithms*, Theoret. Comput. Sci. **92** (1992), no. 2, 319–336.
- [7] Branko Grünbaum, *Arrangements and spreads*, American Mathematical Society Providence, R.I., 1972. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10.
- [8] Ross J. Kang and Tobias Müller, *Arrangements of pseudocircles and circles*, Discrete Comput. Geom. **51** (2014), no. 4, 896–925.
- [9] Johann Linhart and Ronald Ortner, *On the combinatorial structure of arrangements of oriented pseudocircles*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 30, 13 pp. (electronic).
- [10] ———, *An arrangement of pseudocircles not realizable with circles*, Beiträge Algebra Geom. **46** (2005), no. 2, 351–356.
- [11] ———, *A note on convex realizability of arrangements of pseudocircles*, Geombinatorics **18** (2008), no. 2, 66–71.
- [12] Johann Linhart and Yanling Yang, *Arrangements of arcs and pseudocircles*, Beiträge Algebra Geom. **37** (1996), no. 2, 391–398.
- [13] Bojan Mohar and Carsten Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.
- [14] Ronald Ortner, *Embeddability of arrangements of pseudocircles into the sphere*, European J. Combin. **29** (2008), no. 2, 457–469.
- [15] ———, *Forcing subarrangements in complete arrangements of pseudocircles*, J. Comput. Geom. **6** (2015), no. 1, 235–248.
- [16] Carsten Thomassen, *Embeddings of graphs with no short noncontractible cycles*, J. Comb. Theory Ser. B **48** (1990), no. 2, 155–177.